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EXPONENTIALS, THEIR ORIGINS AND DESTINY

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RESUMO. These notes are slightly expanded version of a lecture that I delivered to curious high school students and their math teachers. They were already exposed to the basic notions of calculus and familiar with the formal side of it but still might feel amused by the mysterious "coincidences" like the Euler formula relating exponential function with trigonometry and the reasons the logarithm turned out to be so useful in applications. I argue that it is the differential equations that naturally lead to appearance of these transcendental functions. The final section explains connections between polynomials, differential equations and resonances.

1. WHAT IS AN EXPONENT?

The usual explanation of what is the exponential function $f(x) = \mathrm{e}^x$ of one real variable $x \in \mathbb{R}$ in high school and traditional calculus courses is excruciatingly illogical. Recall that it starts with the definition of powers a^n for an arbitrary positive number $a > 0$ and natural, then integer values of $n \in \mathbb{Z}$. Transition to the rational values (description of a^x for $x \in \mathbb{Q}$) already involves non-arithmetic operation of the root extraction

$$a^{p/q} = \sqrt[q]{a^p}, \qquad p, q \in \mathbb{Z},\ q > 0.$$

The crucial step of extension of the function a^x from $x \in \mathbb{Q}$ to the real line $x \in \mathbb{R}$ is already non-obvious: while $\frac{1}{2} \approx \frac{499}{1000}$, it is not at all clear why

$$\sqrt{2} \approx \sqrt[1000]{2^{499}}.$$

Even if we want to check this on a pocket calculator, the direct calculation of the right hand side will involve numbers larger than $(2^{10})^{50} > 10^{150}$ with more than hundred fifty digits, and because of the rounding errors the results will be quite different. In the high school this question (continuity of a^x on $\mathbb{Q}$) is tacitly swept under the carpet because of various technical reasons (accurate construction of real numbers, undeveloped theory of limits).

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But then in the first year college calculus courses the story becomes even more mysterious. The seemingly arbitrary limit

$$\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = \mathrm{e}$$

is introduced as *Deus ex machina* and certain properties of this limit are established. For instance, the convergence of the seemingly irrelevant series

$$\sum_{k=0}^{\infty}\frac{1}{k!} = \mathrm{e} \tag{1}$$

is proved by sophisticated tricky arguments. And only then it is proved that the derivative $f'(x)$ of the function $f(x) = \mathrm{e}^{\lambda x}$ is equal to $\lambda f(x)$, in particular, if $\lambda = 1$, then $f' = f$.

All this suggests that the cart was put before the horse and the real *raison d'étre* of existence of the exponential function belongs elsewhere. We will show that the exponential is naturally born as a solution to some simple differential equation. The above illogical exposition aims to conceal the fact that differential equations (at least linear ordinary) were known well before Newton and Leibniz while the modern "rigorous" exposition is based on the ideas and notions developed and formalized only in the 19th century (Dirichlet, Weierstrass, Cauchy, ...).

2. What is a differential equation?

The way the derivation is studied in the high school, is largely formal. At the beginning we consider the commutative ring of polynomials $\mathbb{R}[x]$ in one variable and the corresponding field of rational functions $\mathbb{R}(x)$. These algebraic terms mean that the set of all polynomials contains the real numbers $\mathbb{R}$, the reserved symbol for the indeterminate x, and everything which can be produced from them by the three operators $+$, $-$ and the multiplication $\times$, all related by the standard rules of arithmetics. The division (anti-multiplication) is not always possible in $\mathbb{R}[x]$, but if we allow formal fractions $p(x)/q(x)$ with $p, q \in \mathbb{R}[x]$ and $q \neq 0$, then we obtain the field very much like the field $\mathbb{Q}$ is obtained from the ring $\mathbb{Z}$. If we have more than one independent variable $x, y, \dots$, the corresponding notions can be instantly generalized, producing the ring $\mathbb{R}[x, y, z, \dots]$ and the field $\mathbb{R}(x, y, z, \dots)$. If necessary, the ground field $\mathbb{R}$ can be replaced by $\mathbb{C}$, the field of complex numbers.

In the field $\mathbb{R}(x)$ any linear equation $af + b = 0$, $a, b \in \mathbb{R}(x)$, $a \neq 0$, always has a solution. However, already quadratic equation $f^2 = a$, $a \in \mathbb{R}(x)$, may be non-solvable: e.g., there is no rational function $f(x)$ such that $f^2 = x$. This leads naturally to functions expressible in radicals and, more generally, algebraic functions defined by polynomial equations of the form $P(x, f) = 0$, where $P = P(x, y)$ is a polynomial in *two* independent variables.

The derivative of a real function $f(x)$ of one variable x is formally defined as the limit

$$f'(x) = \lim_{y\to x}\frac{f(y) - f(x)}{y - x}, \tag{2}$$

assuming that the right hand side is well defined for $y - x \neq 0$ and the limit exists (which is not true for an arbitrary function f). However, if we differentiate a polynomial $p \in \mathbb{R}[x]$, then one can immediately see that the divided difference

$$\frac{p(y) - p(x)}{y - x}$$

is a *polynomial* $P(x, y)$ *in two variables* x, y. Indeed, for $p(x) = x^n$ this follows from the identity

$$y^n - x^n = (y - x)(y^{n-1} + y^{n-2}x + \cdots + yx^{n-2} + x^{n-1}), \qquad n = 1, 2, \dots$$

A general polynomial is a sum of monomials $a_n x^n$ with constant coefficients and the result follows by additivity. Of course, the polynomial $P(x, y)$ is defined everywhere and in particular on the diagonal $y = x$, where its values give the derivative. Derivation of rational functions is only slightly more difficult. This construction allows to differentiate all functions, even non-polynomial, if they can be represented by converging Taylor (power) series: this fact was Newton's main tool in developing the Mechanics.

As long as we know that derivative of polynomials always exists (and derivative of rational functions exists everywhere in the domain of their definition), we can consider the operation of derivation as a transformation (map) D which associates with any rational function $f = f(x)$ its derivative $Df = f'$. This map has the following properties:

(1) $\mathbb{R}$-linearity: $D(\lambda f + \mu g) = \lambda Df + \mu Dg$ for any two rational functions $f, g \in \mathbb{R}(x)$ and any two constants $\lambda, \mu \in \mathbb{R}$.

(2) Leibniz rule: $D(f \cdot g) = f \cdot Dg + g \cdot Df$ for any two rational functions $f, g \in \mathbb{R}(x)$.

Problem 1. Prove the Leibniz rule for polynomials by the direct computation of the divided differences.

From the formal point of view $D : \mathbb{R}(x) \to \mathbb{R}(x)$ is a *linear operator* over $\mathbb{R}$.

These two rules imply a number of consequences: for instance, $D1 = D(1 \cdot 1) = D1 + D1$, hence $D1 = 0$ and by linearity $D\lambda = 0$ for any $\lambda \in \mathbb{R}$. If $fg = 1$, then $g \cdot Df + f \cdot Dg = 0$, which allows to compute the derivative Dg of the reciprocal function $g = 1/f$: $D(1/f) = -Df/f^2$.

On the other hand, these rules alone *do not define* D *uniquely*: the value Dx cannot be computed using only these rules. However, if Dx is known, the value Df of any other function can be already computed uniquely. The "genuine" derivation corresponds to the case where the extra normalizing condition $Dx = 1$ is added to the above list. By induction one shows then that $Dx^n = nx^{n-1}$ for any $n \in \mathbb{Z}$.

Note that in addition to the arithmetic operations, the ring $\mathbb{R}[x]$ and field $\mathbb{R}(x)$ are closed (with minor caveats for the latter) under the operation of *composition*: one can substitute into any function $g(x)$ instead of x another polynomial or rational expression $h \in \mathbb{R}(x)$, obtaining the new polynomial (resp., function) $p(x) = g(h(x))$ denoted by $p = g \circ h$. This composition is in general non-commutative, $g \circ h \neq h \circ g$!

Problem 2. Prove that the operator D satisfies the chain rule: if $f = g \circ h$, i.e., $f(x) = g(h(x))$, then $Df = [(Dg) \circ h] \cdot Dh$. In particular, if $f(x) = g(\lambda x)$, $\lambda \in \mathbb{R}$, then $Df = \lambda (Dg)(\lambda x)$.

Hint. Because of the linearity of D, it is sufficient to prove the chain rule for the monomial $g = x^n$, that is, prove that $D(h^n) = nh^{n-1} \cdot Dh$ for any rational function $h \in \mathbb{R}(x)$. Recycle the argument with the divided difference. □

Once the operator D is defined, one can consider *differential* equations involving unknown function u and its derivative Du. The simplest equations *linear* in the unknown function u can be progressively listed as follows:

$$Du = b, \qquad b \in \mathbb{R}(x), \tag{3}$$

$$Du = au, \qquad a \in \mathbb{R}(x), \tag{4}$$

$$Du = au + b, \qquad a, b \in \mathbb{R}(x). \tag{5}$$

Of course, this is only the beginning: one can consider general equations of the form $P(x, u, Du) = 0$ defined by a polynomial P in three independent variables and systems of equations of this type. As was the case with algebraic equations, some of them are solvable in rational functions, some not.

The first equation is *almost* solvable. It is solvable for any $b \in \mathbb{R}[x]$: indeed, any equation $Du = x^n$ with $n \in \mathbb{Z}$, $n \geqslant 0$, is solvable: $u(x) = \frac{1}{n+1}x^{n+1}$ (can be verified by the direct computation). Solution for a general polynomial can be obtained by the linearity. The same argument works in fact for all $n \in \mathbb{Z}$ except for $n = -1$. Yet *the equation* $Du = 1/x$ *has no solutions in the rational functions*!

Problem 3. Prove this fact.

Hint. Any rational function $f(x)$ has a converging asymptotic expansion at infinity of the form $f(x) = a_k x^k + a_{k+1} x^{k+1} + \cdots$, $k \in \mathbb{Z}$, $a_k \neq 0$. This expansion can be termwise differentiated. □

Clearly, to develop any reasonable theory of differential equations one has to extend the field $\mathbb{R}(x)$ by this function, in the same way as we extend it by adding radicals when studying algebraic equations.

Remark 4. Solutions of differential equations are almost never unique. Indeed, when solving the equation $Du = f$, we can always add to any solution $u(x)$ any constant $c \in \mathbb{R}$, since $Dc = 0$. This non-uniqueness means that a differential equation, in order to have a uniquely defined solution, must be accompanied by some *initial conditions* (one or several, depending on the order of the equation). For equations of the first order it is usually enough to specify the value of the solution $u(x)$ at a given point $x = a$.

3. Logarithm and antilogarithm

To avoid logical traps, we will provisionally denote by $\ell(x)$ the solution of the equation $Du = 1/x$. In order to avoid the ambiguity with non-uniqueness, we need to specify an initial condition, see Remark 4. It turns out that the conditions

$$D\ell = \frac{1}{x}, \qquad \ell(1) = 0,$$

are the most convenient. The general Newton–Leibniz theorem implies (see Remark 5 below) that ℓ is a continuous function defined for all positive arguments $x > 0$.

These conditions imply certain useful properties of ℓ. For instance, for any constant $\lambda \in \mathbb{R}$ the function $\tilde{\ell}(x) = \ell(\lambda x)$ has the same derivative[1]:

$$D\tilde{\ell} = \lambda(D\ell)(\lambda x) = \lambda(\lambda x)^{-1} = x^{-1}.$$

From this it follows immediately that

$$\ell(\lambda x) = \ell(\lambda) + \ell(x). \tag{6}$$

Indeed, both sides of the equation have the same derivative, hence differ by a constant. It remains only to check that their values coincide at some point, in this case at the point $x = 1$ by virtue of the condition that $\ell(1) = 0$. In other words, the function ℓ considered as an application $\ell : \mathbb{R}_+ \to \mathbb{R}$, *transforms the operation of multiplication of positive numbers to that of addition* (in the algebraic terms, ℓ a homomorphism of Abelian groups $(\mathbb{R}_+, \times)$ and $(\mathbb{R}, +)$. Since multiplication is computationally much heavier operation than addition (not speaking about division!), there is no surprise that such amazing function deserves to be studied, tabulated, portrayed etc.

Remark 5. One can derive the property (6) from the Newton–Leibniz formula. Indeed, the defining equation for ℓ means that

$$\ell(a) = \int_0^a x^{-1}\,\mathrm{d}x,$$

is the area of the trapeze T_a under the graph of the hyperbola $xy = 1$, the horizontal axis and between two vertical lines $x = 1$ and $x = a$ (draw the plot!).

Consider the transformation of the (x, y)-plane into itself which maps a point (x, y) into the point $(bx, b^{-1}y)$ for some positive $b > 0$ (it is called a hyperbolic rotation). Obviously this transformation preserves the hyperbola $xy = 1$ and areas of all figures on the plane. Applying this transformation to the trapeze T_a we get another trapeze under hyperbola, squeezed between the vertical lines $x = b$ and $x = ab$. Its area is $\ell(ab) - \ell(b)$ and it must remain equal to $\ell(a)$. □

Another equation, which has no solution among the rational functions, is the general equation (4). For instance, for any polynomial $a \in \mathbb{R}[x]$ there cannot be a *nontrivial* (different from the identical zero) solution. Indeed, for any polynomial u its degree $\deg u$ is reduced by application of D: $\deg Du = \deg u - 1$, while the degree of the right hand side, $\deg a + \deg u$, is greater or equal to $\deg u$.

Consider first the simplest case where $a = 1$ is a constant (such equation still has no polynomial solutions). It turns out that in this case we can solve this equation using the *function inverse to* ℓ. Let $e(x)$ be the function inverse to ℓ, so that $\ell(e(x)) = x$. Differentiating this identity and using the chain rule again, we conclude that

$$1 = D\ell(e(x)) \cdot De(x) = \frac{1}{(e(x))} \cdot De(x), \quad \text{so that} \quad De = e,$$

and the function $e(x)$ satisfies the equation (4) with $a(x) \equiv 1$.

Since $\ell(x)$ transforms mutliplication into the addition as in (6), it is natural to expect from $e(x)$ to do the opposite:

$$e(\lambda + x) = e(\lambda)e(x). \tag{7}$$

[1]This is assuming that the derivation D satisfies the chain rule not just on $\mathbb{R}(x)$, but also for the non-rational function ℓ.

One can prove this by applying e to both parts of (6), but a direct proof from the equation $De = e$ is more instructive.

First note that $e(x)e(-x) \equiv 1$. Indeed, derivation of the left side yields by the Leibniz rule

$$De(x) \cdot e(-x) - e(x)De(-x) = e(x)e(-x) - e(x)e(-x) = 0,$$

hence the product is a constant. Note that $e(0) = 1$ since $\ell(1) = 0$; this means that the constant $e(x)e(-x)$ is equal to one.

To prove (7), note that together with $e(x)$ any scalar multiple $\tilde{e}(x) = Ce(x)$ satisfies the linear equation $Du = u$, and one can easily see that no other solutions exist (prove it by substituting the product $C(x)e(x)$ into the equation!). Thus $e(x + \lambda) = Ce(x)$. To evaluate the constant C (depending on λ, substitute $x = -\lambda$: since $e(0) = 1$, one must have $C = 1/e(-\lambda) = e(\lambda)$.

The identity (7) implies that for integer values of $x = n$ the function $e(x)$ coincides with the powers e^n, where $\mathrm{e} = e(1)$. The standard argument proves that $e(x) = \mathrm{e}^x$ for all $x \in \mathbb{Q}$. This allows to use the notation e^x also for irrational values of x (where it was lacking an accurate meaning until now).

To solve a slightly more general equation $Du = au$, $a \in \mathbb{R}$ a constant, we can use the function $e(ax) = \mathrm{e}^{ax}$. Prove it!

4. Properties of the exponential function

Now we can study the properties of the function $e(x)$ directly from the differential equation defining it: it will bring us a new information on the function ℓ as well.

Example 6. Instead of the (non-existing) polynomial solution, we can look for a solution of the equation $Du = u$ as a formal Taylor series[2] starting with the free term $c_0 = 1$ to satisfy the initial condition $u(0) = 1$,

$$u(x) = 1 + c_1 x + c_2 x^2 + \cdots + c_n x^n + \cdots .$$

The action of D on such series can be easily extended by linearity:

$$Du = c_1 + 2c_2 x + 3c_3 x^2 + \cdots + (n+1)c_{n+1} x^n + \cdots .$$

Equality of the Taylor series means that the coefficients before the identical powers coincide, so $c_n = (n+1)c_{n+1}$, or better $c_{n+1} = c_n/(n+1)$, $c_1 = 1$. This implies that $c_n = 1/n!$.

A simple study shows that the resulting series

$$e(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!} \tag{8}$$

in fact converges for all $x \in \mathbb{R}$ (compare the growth of factorials with the growth of any geometric progression). In particular, setting $x = 1$ gives a fast converging series (1) for e.

[2] Any reasonable textbook would devote to the notion of a *formal* series a few paragraphs, showing that they constitute a commutative algebra over $\mathbb{R}$ (or $\mathbb{C}$) which extends the algebra of polynomials.

Remark 7. There is a "numerical" way to solve the differential equation $Du = u$, $u(0) = 1$. Rather than looking for a formal series to solve the equation $Du = u$, one can approximate it by the difference equation

$$\frac{u(x+h) - u(x)}{h} = u(x), \qquad h \approx 0,$$

with a given small constant step $h > 0$. This equation can be "integrated forward" (explicitly solved) by a geometric progression:

$$u(x+h) = u(x) + hu(x) = (1+h)u(x).$$

This method, known as the Euler method of numerical integration of differential equations, is very robust and can be shown to converge to the genuine solutions as $h \to 0^+$. In particular, if we start with the initial condition $u(0) = 1$ and set the step $h = 1/n$ for some large $n \gg 1$, then a simple computation shows that

$$u\big(\tfrac{k}{n}\big) = \big(1 + \tfrac{1}{n}\big)^k, \qquad k = 1, 2, \dots.$$

Setting $k = n$, we obtain the Euler formula for the approximation of $u(1)$ in the form $\big(1 + \frac{1}{n}\big)^n$ which, as explained, converges to e as $n \to \infty$.

Remark 8. There exists another, much more efficient method of approximating solutions of the differential equation $Du = u$ with the initial condition $u(0) = 1$. We can rewrite this equation in the integral form

$$u(a) = u_(0) + \int_0^a Du(x)\,\mathrm{d}x = 1 + \int_0^a u(x)\,\mathrm{d}x$$

and consider the right hand side as an operator $I\colon u \mapsto 1 + \int_0^\bullet u\,\mathrm{d}x$. Then we can consider a sequence of "approximate solutions" $u_0, u_1, u_2, \dots$ defined by the recurrent rule $u_{n+1} = Iu_n$, $n = 0, 1, 2, \dots$. If this sequence converges to a limit u_*, then by continuity $u_* = Iu_*$ and hence u_* is the solution of the equation $Du = u$ with the initial condition $u(0) = 1$. The functions u_n are called *Picard approximations* of the solution, and indeed very fast converge to it.

It is a fundamental fact that *integral* operators of the form $u \longmapsto \int_0^\bullet u\,\mathrm{d}x$ are strongly *contracting* in the spaces of continuous functions on $[0, a]$ in the maximum-norm $|u| = \max_{x\in[0,a]} |u(x)|$ if a is small enough. This can be achieved by direct estimates. As a corollary, we conclude that the sequence of Picard approximations fast converges to its limit.

In our specific case the first several Picard approximations can be easily computed:

$$u_0(x) \equiv 1,\ u_1(x) = 1 + x,\ u_2 = 1 + \frac{1}{2}x^2,\ u_3(x) = 1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3,\ \dots,$$

and by induction one can obtain the series (8) for the exponential function, proving at the same time its convergence.

5. What else can be exponentiated?

The ground field $\mathbb{R}$ can be replaced by the field $\mathbb{C}$ without changing anything in the formal settings. The question is how to solve the equation $Du = \lambda u$ for $\lambda \in \mathbb{C}$ and what is e^z for $z \in \mathbb{C}$. In particular, what means the imaginary exponent $\mathrm{e}^{\mathrm{i}x}$ for real values $x \in \mathbb{R}$.

Geometrically the equation $Du = \mathrm{i}u$ describes the motion $x \longmapsto u(x) \in \mathbb{C}$ on the complex plane $z \in \mathbb{C}$, whereby the velocity at the point z is equal to $\mathrm{i}z$ and $x \in \mathbb{R}$ is the real time.

This law of motion means that the velocity $\mathrm{i}u$ is *orthogonal* to the radius-vector $u \in \mathbb{C}$ of the point itself (rotated by $\pi/2$ counterclockwise). By the characteristic property of the circle on the Euclidean plane, its solutions are the circles $|z| = \rho > 0$ run counter-clockwise with the velocity equal to ρ, which means that the period of each rotation is exactly 2π regardless of $\rho > 0$. Taken into account the exponential law, we conclude that the exponential function is periodic in the complex plane,

$$\mathrm{e}^{2\pi\mathrm{i}} = 1, \qquad \mathrm{e}^{z+2\pi\mathrm{i}} = \mathrm{e}^{z} \quad \forall z \in \mathbb{C}.$$

If $u(0) = 1$, then $\rho = 1$ and we have

$$\mathrm{e}^{\mathrm{i}y} = \cos y + \mathrm{i}\sin y, \qquad y \in \mathbb{R}.$$

In other words, the trigonometric functions appear as the real and imaginary parts of the complex solution of the equation $Du = \mathrm{i}u$ with the initial condition $u(0) = 1$.

Remark 9. Of course, one can use the converging series (8) to compute the series

$$\mathrm{e}^{\lambda z} = \sum_{n=0}^{\infty} \frac{\lambda^n}{n!} z^n, \qquad z \in \mathbb{C}, \tag{9}$$

also converging for all $z \in \mathbb{C}$, in particular, for $z = \mathrm{i}y$, $y \in \mathbb{R}$.

However, this is the idea not so great if $y \gg 1$ is a very large number. For instance, if we let $y = 200\pi \approx 600$, then $\mathrm{e}^{200\pi\mathrm{i}} = 1$, but on the way towards computing this unit, one has to add terms of the order of magnitude $600^{600}/600!$ containing more than 250 digits. Periodicity of the complex exponent is a great simplifier!

Another example is the matrix exponential. Indeed, we can consider the differential equation of the form

$$DU = AU, \qquad A \in \mathrm{Mat}(n, \mathbb{C}),$$

where $U = U(x)$ is the unknown matrix function and A a known constant matrix.

If we could define properly the matrix exponent $U = \mathrm{e}^{Ax}$, there is a good chance that it would solve the differential equation. One obvious way to do that is to substitute the matrix A in the series (8) or rather (9):

$$\mathrm{e}^{A} = \sum_{n=0}^{\infty} \frac{1}{n!} A^n.$$

This series makes sense (as any power A^n is well defined) and can be tested to converge absolutely for all A. The formal derivation confirms that thus defined matrix exponent $U(x) = \mathrm{e}^{Ax}$ will be indeed a solution to the equation $DU = AU$.

However, this is not exactly an explicit solution: e.g., given a 2×2-matrix $A = \left(\begin{smallmatrix} \alpha & \beta \\ \gamma & \delta \end{smallmatrix}\right)$, it is a not-too-obvious task to compute the matrix elements of the matrix e^{A} as functions of $\alpha, \beta, \gamma, \delta$.

Obviously, if $A = \mathrm{diag}\{\lambda_1, \dots, \lambda_n\}$ is a diagonal matrix, then all powers of $A^k = \mathrm{diag}\{\lambda_1^k, \dots, \lambda_n^k\}$ are also diagonal and the exponential series can be instantly computed:

$$A = \begin{pmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \end{pmatrix} \implies \mathrm{e}^{A} = \begin{pmatrix} \mathrm{e}^{\lambda_1} & & & \\ & \mathrm{e}^{\lambda_2} & & \\ & & \ddots & \\ & & & \mathrm{e}^{\lambda_n} \end{pmatrix}.$$

We can try to transform the equation $DU = AU$, by, say, change of variables of the form $U = VC$ or $U = CW$ with, say, a constant invertible matrix C. The first transformation does not change the equation: since $D(VC) = (DV)C + V(DC)$ and $DC = 0$, we conclude that $(DV)C = AVC$ and V satisfies the same equation. On the other hand, the left multiplication changes the equation:

$$D(CW) = C(DW) = ACW \implies DW = (C^{-1}AC)W.$$

In other words, such a transformation reduces the equation $DU = AU$ to the equation $DW = BW$ with $B = C^{-1}AC$. Such matrix B is called *similar* to the matrix A.

The transformation of similarity

$$A \longmapsto C^{-1}AC$$

sometimes allows to *diagonalize* A, for instance, in the case where all eigenvalues $\lambda_1, \dots \lambda_n$ of A are pairwise different. Yet for matrices with *multiple eigenvalues* this is not always possible: e.g., the nonzero matrix $\left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right)$ with $\lambda_1 = \lambda_2 = 0$ is clearly non-similar to the null 2×2-matrix similar only to itself.

Still one can always explicitly construct a matrix C such that $C^{-1}AC$ is in the *Jordan normal form* J. The simplest way[3] to describe this form is as follows:

(1) $J = \Lambda + N$, where Λ is a diagonal matrix and N is an upper triangular matrix whose nth power is zero: $N^n = \mathbf{0}$ (here n is the dimension of the matrices in question and $\mathbf{0}$ is the zero matrix);
(2) Λ and N *commute*: $\Lambda N = N\Lambda$.

The second property allows to substitute Λ and N instead of a and b in the identity $\mathrm{e}^{a+b} = \mathrm{e}^a \cdot \mathrm{e}^b$ expanded in the Taylor series[4] and prove that

$$\mathrm{e}^{\Lambda+N} = \mathrm{e}^{\Lambda} \cdot \mathrm{e}^{N}.$$

The exponent of the diagonal term was already computed. The computation of the exponent e^N reduces to the computation of the polynomial (the truncation of the exponential series at the nth term). Together this proves the central result of the standard undergraduate course on differential equations.

Theorem 10. *Each matrix element of the matrix exponent e^{Ax} is a linear combination of the terms of the form $x^k \mathrm{e}^{\lambda x}$, $k < n$, λ being an eigenvalue of A and $k \in \mathbb{Z}_+$ an integer strictly smaller than the multiplicity of λ.*

Remark 11. Recall that reducibility to the Jordan normal form holds only over $\mathbb{C}$: if A is a real $(n \times n)$-matrix, then there may be no real matrix C such that $C^{-1}AC = J$.

Remark 12 (short digression, can be skipped at the first reading)**.** There is a general question of which functions of matrices can be defined and when. For instance, it would not be illogical to define $f(A)$ as a diagonal matrix with entries $f(\lambda_k)$ if $A = \operatorname{diag}\{\lambda_1, \dots, \lambda_n\}$, but how useful this generalization would be? Another, more natural solution would be to consider only analytic functions $f(z)$ which are converging Taylor series and substitute A into these series instead of the scalar variable z, as we did with the exponential series. However, if f is a function which is not entire (say, the logarithm $\ln z$ which has a ramification point at $z = 0$), we have problems.

[3] Instead of specifying various blocks, diagonal and upper-triangular, as it is usually done in the standard courses of linear algebra. Check that this is indeed the equivalent description!

[4] Unfortunately, if A and B do not commute, then $\mathrm{e}^{A+B} \neq \mathrm{e}^A \mathrm{e}^B$ in general.

The alternative is to generalize the Cauchy integral formula. Recall that for a function $f(z)$ analytic in a domain $S \subseteq \mathbb{C}$, we have the integral representation

$$f(z) = \frac{1}{2\pi \mathrm{i}} \oint_{\partial S} \frac{f(\zeta)}{\zeta - z} \,\mathrm{d}\zeta.$$

Here ∂S is the boundary of S.

It turns out that one can substitute a matrix A into this formula, and it remains true if the domain S contains all eigenvalues of the matrix A. The result of the substitution in the right hand side is the integral

$$\frac{1}{2\pi \mathrm{i}} \oint_{\partial S} f(\zeta)(\zeta - A)^{-1} \,\mathrm{d}\zeta,$$

where $(\zeta - A)^{-1}$ is the inverse to $\zeta E - A$, which is a holomorphic matrix function outside the set of eigenvalues of A. The integral can be computed by the method of residues, which amounts to the computation of the residues of the matrix elements of the matrix function $f(\zeta)(\zeta - A)^{-1}$. If all eigenvalues of A are simple, then the residues are linear combinations of $f(\lambda_k)$, that is, the exponents of the eigenvalues if $f(z) = \mathrm{e}^z$.

Arguing this way, one can prove that any non-degenerate matrix A has matrix logarithm $\ln A$: the equation $\mathrm{e}^X = A$ is solvable if and only if $\det A \neq 0$, i.e., A has no null eigenvalues. For this it is sufficient to slit the complex plane along a ray $\mathbb{R}_+\mu \subseteq \mathbb{C}$ not passing through any eigenvalue of A, and choose a closed path encircling all these eigenvalues. The slit can be done in a number of non-equivalent ways, which means that the matrix logarithm is always multivalued.

6. Stability

Though different representations via exponentials give in a sense explicit closed form solutions to differential equations, one has still to do an extra work to derive from these explicit expressions their properties important for applications.

The matrix exponentiation corresponds to solution of a system of first order differential equations: if $A = \big(a_{ij}\big)$, then the system has the form

(10) $$Du_i = \sum_{j=1}^{n} a_{ij} u_j, \qquad i = 1, \dots, n, \quad a_{ij} \in \mathbb{R}.$$

Its solutions $u_i = u_i(x)$ are, as we already know, linear combinations of terms of the form $x^k \mathrm{e}^{\lambda_j x}$ corresponding to all eigenvalues $\lambda_1, \dots, \lambda_n$ of A.

For various applications (engineering, chemistry, biology, ...) it is of paramount importance to know the *stability* of the system. There are several closely related but not identical definitions of stability. For linear systems, however, all of them coincide and can be given in the most simple terms.

Definition 13. A system of linear equations (10) is *stable*, if all its solutions remain bounded in the absolute value as $x \to +\infty$. Otherwise the system is called *unstable*.

Theorem 10 gives us the description of solutions as sums of quasimonomials $x^k \mathrm{e}^{\lambda x}$, whose limit behavior as $x \to +\infty$ is well known:

(1) Each term $\mathrm{x}^k e^{\lambda x}$ with $\operatorname{Re}\lambda < 0$ tends to zero as $x \to +\infty$ and to infinity if $\operatorname{Re}\lambda < 0$ regardless of k.
(2) in the intermediate case $\operatorname{Re}\lambda = 0$ solution remains bounded if $k = 0$ and tends to infinity if $k > 0$.

Note that the term $x^k e^{\lambda x}$ with $k > 0$ can occur only if λ is a multiple eigenvalue of A. This implies the following description of stability of systems.

Theorem 14. *Let $\lambda_1, \dots, \lambda_n \in \mathbb{C}$ be eigenvalues of the matrix A, listed with multiplicity.*

(1) *If there exists an eigenvalue with $\operatorname{Re} \lambda > 0$, then the system is unstable.*

(2) *If all eigenvalues satisfy the condition $\operatorname{Re} \lambda \leqslant 0$, then the system is:*

   (a) *stable in the* non-resonant case *where all "boundary eigenvalues" with $\operatorname{Re} \lambda = 0$ are simple,*

   (b) *generically*[5] *unstable in the resonant case, where there is a multiple eigenvalue with $\operatorname{Re} \lambda = 0$.*

In other words, to determine the stability in the critical case where all eigenvalues have zero real parts, one has to reduce the matrix A to its Jordan normal form. If it is diagonal (which is guaranteed in absence of multiple eigenvalues), then the system is stable. If there are nontrivial blocks of size $\geqslant 2$, the system is unstable.

Remark 15. In the real case where the matrix A has real entries, the "critical" eigenvalues appear in the form of complex conjugate pairs $\lambda = \pm i\omega$, $\omega > 0$. The corresponding "exponential" real solutions are $\sin x$ and $\cos x$, obviously bounded. If such eigenvalues are multiple, in addition to these solutions we will (in general) have solutions of the form $x \sin x$ and $x \cos x$, clearly unbounded.

7. A CONCURRENT THEORY: HIGHER ORDER EQUATIONS WITH CONSTANT COEFFICIENTS

7.1. **Other linear equations of first order.** We can complete now the discussion about linear equations of the first order. They can be explicitly solved if we allow operations of exponentiation and antiderivation (computation of primitives) as admissible (cf. with solution of algebraic equations in radicals).

Indeed, the homogeneous equation $Du = au$ with general (non-constant) rational function $a \in \mathbb{R}(x)$ can be solved by a substitution $u = e^v$ which transforms it to the form (3): indeed,

$$e^v \cdot Dv = ae^v \implies Dv = a.$$

In the standard notation this yields

$$u(x) = Ce^{\int^x a(s)\,ds}.$$

The non-homogeneous equation $Du = au + f$ with $a, f \in \mathbb{R}(x)$ can in turn be reduced to a homogeneous equation $Dv = av$ (see above) and yet another antiderivation. Indeed, if v satisfies the homogeneous equation, substitute $u = wv$. Then

$$Dw \cdot v + w \cdot Dv = avw + f$$

Since $w(Dv - av) = w0 = 0$, we conclude that w satisfies the equation of the type (3)

$$Dw = v^{-1} f.$$

[5] Exceptional stability occurs when there is a multiple eigenvalue, but the Jordan form of A restricted on the corresponding invariant subspace is diagonal without nontrivial upper-triangular blocks.

Note that in both cases the function $v = \int^x a(s)\,\mathrm{d}s$ in general is not rational (remember the logarithms!), so the primitive (antiderivation) should be applied to a more complicated function (still elementary, if $a(x)$ was rational).

7.2. **Homogeneous equations of higher order.** In parallel with systems of first order equations (or equivalent matrix differential equations), there is a natural class of *higher order differential equations*. They resemble the matrix equations albeit exhibiting certain specific features.

Example 16. The equation $D^r u = 0$, $r \in \mathbb{N}$, has r-dimensional space of solutions: all polynomials of degree $\leqslant r-1$.

The equation $(D-\lambda)^r u = 0$ can be reduced to the previous case by the substitution $u = v\mathrm{e}^{\lambda x}$. Indeed, then $(D-\lambda)u = \mathrm{e}^{\lambda x} Dv$, that is, on the level of *differential operators* we have the identity

$$(D-\lambda)\mathrm{e}^{\lambda x} = \mathrm{e}^{\lambda x} D.$$

Denote by L the "zero order differential operator" $u \mapsto Lu = \mathrm{e}^{\lambda x}u$. It is invertible: L^{-1} corresponds to multiplication by $\mathrm{e}^{-\lambda x}$, and the above identity then reads $L^{-1}(D-\lambda)L = D$. "Multiplying" (in fact, composing) this identity with itself r times, we conclude that

$$L^{-1}(D-\lambda)^r L = D^r.$$

This transforms the equation $(D-\lambda)^r u = 0$ to the equation $D^r v = 0$. Note the similarity of this transformation with the similarity of invertible matrices in the group $\mathrm{GL}_n(\mathbb{C})$.

Solutions of the equation $(D-\lambda)^r u = 0$ are therefore *quasipolynomials*, functions of the form $u(x) = x^k \mathrm{e}^{\lambda x}$, $k = 0, 1, \ldots, r-1$.

Consider now the general homogeneous equation with constant coefficients of the form

$$D^r u + a_1 D^{r-1} u + \cdots + a_{r-1} Du + a_r u = 0, \qquad a_i \in \mathbb{C}. \tag{11}$$

This equation can be written under the form $P(D)u = 0$, where $P \in \mathbb{C}[\zeta]$ is the corresponding polynomial in one formal variable ζ with complex coefficients.

The polynomial P factors as the product of powers of different linear factors,

$$P(\zeta) = \prod (\zeta - \lambda_i)^{r_i}, \qquad \lambda_i \neq \lambda_j, \quad r_i \geqslant 1, \ \sum r_i = r$$

Correspondingly, the linear equation can be written as the composition of *commuting* differential operators $(D-\lambda_i)^{r_i}$. Since the terms can be reordered, any of these powers can be placed at the rightmost side, hence solution of the equation contains solutions of each equation $(D-\lambda_i)^{r_i} u = 0$, solved above. These solutions are obviously linear independent over $\mathbb{C}$ and together give a fundamental set of solutions to (11) of dimension $r = \sum r_i$.

Remark 17. Note that in the case of equations the resonance phenomenon is not merely possible, like in the case of matrix equations, but inevitable: among solutions there always will be terms of the form $x^k \mathrm{e}^{\lambda x}$, if λ is a multiple root of the polynomial P.

7.3. **Non-homogeneous equations.** A non-homogeneous equation

$$P(D)u = f \tag{12}$$

can be reduced to solution of the homogeneous equation $P(D)u = 0$ and integration. This will give an explicit set of solutions in the form of Liouvillean functions (rational functions, exponents and their primitives). However, there is a more direct way to study the problem when the nonhomogeneity f is of a special form, namely, itself is a solution of a linear equation with constant coefficients. The typical case is the function of the form $\sin \omega t$ which is a solution of the equation $(D^2 + \omega^2)f = 0$.

Let $Q(D)$ be the minimal order linear operator with constant coefficients which annuls f. Then, applying Q to both parts of the equation $P(D)u = f$, we obtain the linear homogeneous equation $R(D)u = 0$, where $R = PQ \in \mathbb{C}[\zeta]$ is the product. Solutions of this equation give simultaneously all solutions to all equations of the form $P(D)u = g$ for all non-homogeneities g which are *conjugate* with f (solving the same equation as f does).

Remark 18. Non-homogeneous equations appear in the so called *system theory*, in which the homogeneous equation $P(D)u$ describes a system (mechanical, electric, biological) and the right hand side f is interpreted as the *input*. If the system is initially unstable, or the input is unbounded, then the non-homogeneous equation obviously has unbounded solutions. The question is whether a bounded input can produce unbounded *reaction* of stable system.

7.4. **Resonances.** We consider the non-homogeneous equation (12) with the input f satisfying a linear homogeneous equation $Q(D)f = 0$. As explained above, the issue of stability reduces to description of the roots of the product polynomial $PQ \in \mathbb{C}[\zeta]$.

If $\gcd(P, Q) = 1$, that is, the polynomials Q and P have no common roots, then the spectrum (the set of roots counted with their multiplicities) of PQ is the union of spectra of P and Q, hence the non-homogeneous equation is stable **if and only if** the homogeneous equation (system) is stable and the input is bounded.

Example 19. Assume that $f = ce^{\mu x}$ which corresponds to $Q = D - \mu$. Then the equation $PQ(D)u = 0$ reduces to the equation $(D - \mu)u = v$, where v is a solution of the homogeneous equation $P(D)v = 0$. This can be done explicitly, as explained in §7.1.

Remark 20. The above example suggests how to find a particular solution in the case where f is a finite linear combination of exponents different from the roots of the equation $P(\zeta) = 0$. The next natural step is to consider infinite sums of the form $f(x) = \int c(\zeta)\mathrm{e}^{\zeta x}\,\mathrm{d}\zeta$. A systematic development of this approach yields the Laplace method of solving linear ordinary differential equations.

If, on the contrary, $\gcd(P, Q) \neq 1$, then R will have necessarily *multiple roots*, which can produce the resonance as explained in §7.2. This is especially important when the common roots form conjugate pairs $\pm \mathrm{i}\omega_k$. The values $\omega_k > 0$ are called (critical) frequencies.

Theorem 21. *If the homogeneous equation and the equation for inputs have **common frequencies**, then the non-homogeneous system is unstable: there are unbounded reactions corresponding to bounded inputs.*

Remark 22. In the analytic theory of differential equations, developed by efforts of many great mathematicians in the 20th century, from Poincaré and Dulac to Siegel, Kolmogorov, Arnold and Moser, the notion of resonance was understood much more deeply as an arithmetic identity between eigenvalues of appropriate linear operators. Unfortunately, this subject goes way aside from the modest main theme of this lecture.

8. Concluding remarks

The elementary theory described above can be described as *commutative*: indeed, it replaces all calculations with differential equations by calculations in the algebra of polynomials or, at worst, in the commutative subalgebras of the non-commutative algebra of matrices.

To develop the general theory, one has to generalize the corresponding calculus to the case of *non-commutative polynomials*. One possibility is the *Weyl algebra* in two variables X, D, where X is interpreted as the operator of multiplication by the independent variable,

$$X : u(x) \mapsto xu(x).$$

The Leibniz rule allows to compute the *commutator* $[D, X] = DX - XD$:

$$\begin{aligned}(DX - XD)u &= D(xu) - xDu \\ &= u + xDu - xDu = u, \implies DX - XD = 1.\end{aligned}$$

Using this commutator identity one can reduce any polynomial to the form $L(X, D) = \sum_k a_k(X)D^k$ and study the possibility of factorization of such polynomials. This theory is very rich and exhibits many fascinating features.

9. Further reading

V. I. Arnold, *Ordinary Differential Equations*. Universitext. Springer-Verlag, Berlin, 2006. ii+334 pp.

Yu. Ilyashenko, S. Yakovenko, *Lectures on analytic differential equations*. Graduate Studies in Mathematics, 86. American Mathematical Society, Providence, RI, 2008. xiv+625 pp.

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