RMU
REVISTA MATEMÁTICA UNIVERSITÁRIA

SBM
SOCIEDADE BRASILEIRA DE MATEMÁTICA

**Revista Matemática Universitária, vol. 2, 2020**

**ISSN:** 2675-5254 – **DOI:** https://doi.org/10.21711/26755254/rmu202024

# "SIMPLIFICATION" IN PARTIAL DIFFERENTIAL EQUATIONS

CARLOS KENIG

Fourier analysis and partial differential equations have been intertwined since their beginnings in the work of J. Fourier (1768–1830). In his study of heat conduction [15], [16], Fourier derived an equation (the heat equation) to describe the heat flow in a one dimensional bar, from Newton's law of cooling. The equation [15] is

$$\partial_t u - \partial_x^2 u = 0, \tag{1}$$

where $u$ is the temperature. Fourier solved the "initial value problem" (IVP). Thus, given the initial temperature $u_0$ of an infinite bar (0 near spatial infinity), Fourier calculated the future temperature $u$ at any point in the bar. In order to do this, Fourier made the following claim: given any function $u_0$ on $\mathbb{R}$ (0 near infinity), it can be represented as an integral of trigonometric functions

$$u_0(x) = \int_{-\infty}^{+\infty} e^{2\pi i x\cdot\xi} c(\xi)\, d\xi, \tag{2}$$

where

$$c(\xi) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} e^{-2\pi i x\cdot\xi} u_0(x)\, dx. \tag{3}$$

We now denote $c(\xi) = \widehat{u}_0(\xi)$, the Fourier transform of $u_0$.

This claim by Fourier was very controversial: $\{e^{2\pi i x\cdot\xi}\}$ are "good functions", so how can it be that "bad functions" can be represented as their integral? Fourier's basic idea is that there is a "simplification" in solving (IVP), and one only needs to solve for $u_0(x) =$

---

Data de aceitação: 15 de julho de 2020.

*Palavras chave.* Partial Differential Equations.

Supported in part by NSF Grant DMS-1800082.

This note is based on the author's lecture at AIM, the African Institute of Mathematics in Cape Town, South Africa, on February 24, 2020, during the "Day of Engagement" organized by the South African Mathematical Society. The author thanks the participants and the hosts of this event for their interest and their warm hospitality.

$e^{2\pi i x\cdot\xi}$ (plane waves). One then obtains the general solution by integration. For the heat equation, if $u_0(x) = e^{2\pi i x\cdot\xi}$, then $u(x,t) = e^{2\pi i x\cdot\xi} \cdot e^{-t4\pi^2|\xi|^2}$.

Even though Fourier’s claim is false as stated, for all $u_0$ and all points $x$, Fourier’s method worked, and yielded important technological applications. Here is a sample of them, all due to W. Thompson (Lord Kelvin) ([3], [36], [39])

- The construction of the transatlantic cable (1855).
- The calculation of the temperature of the Earth (1869).
- The calculation of the age of the Earth (1869).
- The prediction of tides (1876).
- The “harmonic analyzer”, a mechanical device that computed Fourier transforms and thus predicted tides (1876).
- etc.

The Fourier representation (2), which holds for many functions $u_0$, is still a topic of intensive investigation. It was instrumental in the spectacular development of linear partial differential equations in the 20th century. Nevertheless, linearity was essential, and the natural world is mostly nonlinear.

Now we turn to a class of nonlinear equations, the dispersive ones, that appear in nonlinear phenomena of wave propagation. For those equations there is also a “simplification”, this time asymptotically in time. Nonlinear dispersive equations are time reversible, unlike the heat equation, and in the linear setting they can be solved by Fourier’s method. Typically, they have a conserved energy, which may or may not have a sign. These equations model phenomena of wave propagation coming from physics and engineering. Some of the areas that give rise to these equations are water waves, lasers and nonlinear optics, nonlinear elasticity, ferromagnetism, particle physics, general relativity and as geometric flows in Kähler and Minkowski geometries. These equations have been studied extensively in the last 40 years, but they were first introduced in the 19th century. This is currently one of the most exciting areas of partial differential equations. Here are some examples:

**a):** The (generalized) Korteweg–de Vries equation $(gKdV)_k$ ($KdV$ corresponds to $k = 1$) (models water waves in shallow channels)

$$\begin{cases} \partial_t u - \partial_x^3 u + u^k \partial_x u = 0 & x \in \mathbb{R}, t \in \mathbb{R}, \\ u|_{t=0} = u_0, & k = 1, 2, 3, \dots \end{cases} \tag{4}$$

**b):** Nonlinear Schrödinger equations (NLS) (model nonlinear optics, lasers, ferromagnetism, quantum field theory)

$$\begin{cases} i\partial_t u + \Delta u \pm |u|^{p-1} u = 0 & x \in \mathbb{R}^N, t \in \mathbb{R}, \\ u|_{t=0} = u_0, & p > 1. \end{cases} \tag{5}$$

**c):** Nonlinear wave equations (NLW) (models for nonlinear elasticity, particle physics, general relativity)

$$\begin{cases} \partial_t^2 u - \Delta u \pm |u|^{p-1} u = 0 & x \in \mathbb{R}^N, t \in \mathbb{R}, \\ u|_{t=0} = u_0, \partial_t u|_{t=0} = u_1, & p > 1 \end{cases} \tag{6}$$

In both (5) and (6), $p = \frac{N+2}{N-2}$ is the "energy critical" case. The $+$ sign corresponds to the "defocusing" case, while the $-$ sign corresponds to the "focusing" case.

**d):** Wave maps (WM), a geometric example (models particle physics, $\sigma$-models, general relativity).

$$u : \mathbb{R}^N \times \mathbb{R} \longrightarrow S^N \hookrightarrow \mathbb{R}^{N+1},$$

where $S^N$ is the unit sphere.

$$\begin{cases} \partial_t^2 u - \Delta u - [|\nabla u|^2 - (\partial_t u)^2]u = 0, \\ u|_{t=0} = u_0,\ \partial_t u|_{t=0} = u_1. \end{cases}$$

The "energy critical" case is $N = 2$.

These nonlinear equations are called "dispersive" because their linear parts are dispersive. Heuristically, linear evolution equations are dispersive, when the flow "spreads out" or "disperses" the initial data. Since energy is conserved, the size of the linear solution has to become small for large time. This is called the "dispersive effect". In the nonlinear versions, there may be solutions that evolve non–dispersively, like static solutions or solitons (traveling wave solutions). They are "nonlinear objects". Their existence has been a "mystery" since the $19^{\text{th}}$ century. John Scott Russell, a Scottish engineer, first saw such a traveling wave propagating in a canal, in 1834, and chased it on horseback [28]. His observations were viewed with skepticism by Airy and Stokes, since their linear water wave theories could not explain them. Rayleigh and Boussinesq put forward nonlinear theories to explain Russell's observations. Finally, in 1895 [24], Korteweg and de Vries formulated the $KdV$ equation $(gKdV)_1$ in (4). However, its fundamental properties were not understood until much later.

In the late 70's and 80's many properties of nonlinear dispersive equations were discovered, notably the existence and stability (mostly conditionally) of traveling waves. In the late 80's and early 90's, Kenig–Ponce–Vega introduced (see [23] for instance) the systematic use of the machinery of modern Fourier analysis to study the associated linear problems, which was then used perturbatively to study the associated nonlinear problems. The resulting body of techniques, with refinements and extensions by Bourgain, Tao, Tataru, Klainerman-Machedon and many others, proved extremely powerful. Thus, satisfactory theories were obtained for the "short time well-posedness" and for the "global in time well-posedness for small data". A notion of "criticality" linked to "scaling" emerged then.

The last 20 years have seen a shift in emphasis to the study of the long-time behavior of large solutions. Issues like blow-up, global existence and scattering came to the forefront, especially in critical problems. This study was transformed by work of Kenig–Merle ([20], [21], [22], etc) in the period 2005–2009, who introduced the "concentration–compactness/rigidity theorem" method, which has now become the standard approach to this problematic. The ultimate goal of the Kenig–Merle program was to attack the problem of "asymptotic soliton resolution".

Since the 1970's there has been a widely held belief in the mathematical physics community that "coherent structures" and "free radiation" describe the long-time asymptotic behavior of solutions to nonlinear dispersive equations. This came to be known as the "soliton resolution conjecture". Roughly speaking, this conjecture says that, asymptotically in time, for nonlinear dispersive equations, the evolution decouples as a sum of (modulated) traveling wave solutions and a free radiation term (that is a dispersive term solving the associated linear equation). This is a remarkable, beautiful claim, which shows an "asymptotic simplification" in the complex, long–time dynamics of general solutions. The origin of this conjecture goes back to the puzzling numerical simulation of Fermi–Pasta–Ulam [38] at Los Alamos in the early 1950's. This was the birth of scientific computation. Fermi thought that a good use of "The Maniac", the computer built at Los Alamos to do the calculations for the Manhattan Project, was to apply it to a theoretical scientific purpose. He proposed a numerical experiment (the first numerical simulation of a partial differential equation) to verify the principle of "thermalization". Thus, Fermi–Pasta–Ulam considered a vibrating string, with a quadratic nonlinearity, modeled by discretization on a lattice. "Thermalization" meant that if one considered data of energy concentrated in one mode, the effect of the evolution should be to equidistribute the energy among all modes. This was what Fermi expected. Surprisingly, the simulation showed that this need not happen. Fermi called this "a minor discovery" (see [38]). Fermi died soon after and this remained a mystery for many years. In the 1960's, M. Kruskal [25] made the observation that if one lets the mesh of the lattice tend to 0, the discretization of the vibrating string, with quadratic nonlinearity, converges to the $KdV$ equation (4) above, with $k = 1$. As was first observed by Russell on his horse in 1834, the $KdV$ equation has traveling wave solutions, and from those it is obvious that "thermalization" cannot take place.

In 1965, Kruskal-Zabusky [40] carried out another very influential numerical simulation on $KdV$ and discovered two remarkable facts:

**a):** For simple data concentrated on one mode, the evolution, for large time "equals" a sum of traveling waves.

**b):** If the data is a sum of two traveling waves, through the evolution the two traveling waves eventually collide, and after the collision, they reappear unchanged, except for translation ("elastic collision").

a) was performed in connection with the Fermi–Pasta–Ulam paradox. It directly gave rise to the "soliton resolution conjecture". b), in retrospect, was a consequence of "integrability", an important new concept in nonlinear science, that emerged from this simulation (see [41]). Integrable nonlinear equations can be reduced to a collection of linear problems. It is a very non-generic phenomenon, but remarkably ubiquitous. For the $(gKdV)_k$ equations in (4), it holds for $KdV$ $(k = 1)$ or for $(mKdV)(k = 2)$, the modified $KdV$ equations, but not for other $k$. "Soliton resolution" has been proved in a few integrable cases, like $k = 1, 2$ in $(gKdV)_k$, and $p = 3, N = 1$ in (NLS) (5) above ([27], [28], [38], [2]). The last case is the only integrable case of (NLS) (see [30], [31], [2]). The proofs use the method of "inverse scattering" [41]. Even in those cases, the proofs are challenging and issues are still unresolved.

There have also been many results in perturbative regions (near traveling waves). However, the quest to establish "soliton resolution" in the large, for non–integrable models which are time reversible and with conserved energies (Hamiltonian) has been a grand challenge in partial differential equations for the past 50 years [35]. It remained open despite many attempts (see [33], [34] for partial results on (NLS) in high dimensions). The impasse was broken in the work of Duyckaerts–Kenig–Merle [6], on the 3-d radial energy critical wave equation. In the work [6], the authors gave a mathematical quantification of the mechanism observed numerically and experimentally, yielding relaxation to a "coherent structure". This is the radiation of the excess energy to spatial infinity, which appears in such diverse settings as the dynamics of gas bubbles in a compressible fluid and in the formation of black holes in gravitational collapse.

Thus, to be more concrete, consider, for $x \in \mathbb{R}^3$ $(\mathbb{R}^N)$, $t \in \mathbb{R}$,

$$\text{(NLW)} \qquad \begin{cases} \partial_t^2 u - \Delta u = u^5 \ (|u|^{\frac{4}{N-2}} u) \\ u|_{t=0} = u_0 \in \dot{H}^1(\mathbb{R}^3) \ (\mathbb{R}^N) = \{u_0 : \nabla u_0 \in L^2\} \\ \partial_t u|_{t=0} = u_1 \in L^2(\mathbb{R}^3) \ (\mathbb{R}^N) \end{cases}$$

This is a "focusing" equation, i.e. the linear and nonlinear terms have opposite signs, so they "fight each other". This is an energy critical equation, that is, for $\lambda > 0$, the linear and nonlinear terms have the same "strength" under scaling. For $\lambda > 0$,

$$u_\lambda(x,t) = \frac{1}{\lambda^{1/2}} u(x/\lambda, t/\lambda) \ \ (\frac{1}{\lambda^{\frac{N-2}{2}}} u(x/\lambda, t/\lambda))$$

is also a solution, and

$$||(u_{0,\lambda}, u_{1,\lambda})||_{\dot{H}^1 \times L^2} = ||(u_0, u_1)||_{\dot{H}^1 \times L^2}.$$

The equation has a conserved energy, namely

$$\begin{aligned} E(u_0, u_1) = & \frac{1}{2} \int |\nabla u_0|^2 + \frac{1}{2} \int u_1^2 - \frac{1}{6} \int u_0^6, \\ & (-\frac{N-2}{2N} \int |u_0|^{\frac{2N}{N-2}}) \end{aligned}$$

is preserved by the evolution. This equation has solutions that don't disperse, for example solutions to the elliptic equation

$$\Delta Q = Q^5 \ \ (|Q|^{\frac{4}{N-2}} Q), \ Q \in \dot{H}^1, \ Q \not\equiv 0, \tag{7}$$

are non-zero static solutions to (NLW). The one of least energy is

$$W(x) = (1 + \frac{|x|^2}{3})^{-\frac{1}{2}},$$

or for general $N$,

$$W(x) = (1 + \frac{|x|^2}{N(N-1)})^{-\frac{N-2}{2}},$$

up to sign, scaling and translation (for $\lambda > 0, x_0 \in \mathbb{R}^N$,

$$\pm W_{\lambda, x_0}(x) = \pm \frac{1}{\lambda^{\frac{N-2}{2}}} W(x - x_0/\lambda)$$

(see [1], [32]).

There are also traveling wave solutions to (NLW). They are obtained by Lorentz transformation of static solutions. For instance, in $\mathbb{R}^3$, if $\vec{l} \in \mathbb{R}^3$, $|\vec{l}| < 1$, and $Q$ is a non-zero static solution, $Q_{\vec{l}}(x,t) = Q_{\vec{l}}(x - \vec{l}t, 0)$ is a traveling wave solution, where $Q_{\vec{l}}(x,0) = Q(\frac{x_1}{1-l^2}, x_2, x_3)$, when $\vec{l} = l(1,0,0), |l| < 1$. In fact, Duyckaerts–Kenig–Merle [7] have proved that these are all the traveling solutions of (NLW). Moreover, from [26], [17], it follows that, in the radial case (so that there are no traveling wave solutions), the only static solutions are

$$\pm W_{\lambda,0}(x) = \pm\frac{1}{\lambda^{\frac{N-2}{2}}}W(x/\lambda).$$

These models are non-integrable.

We now turn to the main result in [6].

**Theorem 1.** *([6]) Let $u$ be a radial solution of (NLW), $N = 3$, which exists for $t \geq 0$. Then there exists $J \in \mathbb{N}\cup\{0\}$, a radial solution $v_L$ of the linear wave equation ($\partial_t^2 v_L - \Delta v_L = 0$), and for all $1 \leq j \leq J, i_j \in \{\pm 1\}$ and $\lambda_j(t) > 0, 0 < \lambda_1(t) \ll \lambda_2(t) \ll \ldots \ll \lambda_J(t) \ll t$ (where $\ll$ means that the ratio goes to $0$ as $t \to \infty$) such that*

$$u(t) = \sum_{j=1}^{J}\frac{ij}{\lambda_j(t)}W(x/\lambda_j(t)) + v_L(t) + o(1)$$

$$\partial_t u(t) = \partial_t v_L(t) + o(1),$$

*where $o(1)$ denotes a term which tends to $0$ in $\dot{H}^1$ as $t \to \infty$ (respectively in $L^2$).*

The key step in the proof is an “energy channel estimate” for radial solutions of (NLW), $N = 3$:

Let $u$ be a radial solution of (NLW), when $N = 3$, which exists for all $t \in \mathbb{R}$. Assume that $u \not\equiv 0$, and $u \not\equiv \pm W_\lambda$. Then, there exist $R > 0,\ \eta > 0$, such that, for all $t \geq 0$ or for all $t \leq 0$,

$$\int\limits_{|x|\geq|t|+R} |\nabla_{x,t}u|^2 dx \geq \eta. \tag{8}$$

The estimate (8) quantifies the “radiation of excess energy to spatial infinity” (see [6]). However, the analog of (8) fails in the radial case for $N > 3$, and in the non-radial case, for $N \geq 3$. In spite of this, some weaker results were obtained, using monotonicity after time averaging, in analogy with some geometric flow problems. The results obtained in this way, instead of holding for all times, hold for “well-chosen” sequences of times, in the spirit of Tauberian arguments (see [5] for the use of this technique in a result that preceded Theorem 1).

**Theorem 2.** *([27], [3], [19]) The analog of Theorem 1 holds for radial solutions of (NLW), $N > 3$, but only for a “well-chosen” sequence of times $\{t_n\},\ t_n \to \infty$, instead of for all $t \to \infty$.*

[27] deals with all odd $N, N > 3$, [3] with $N = 4$ and [19] with all even $N$. A similar result holds in the non-radial case, $3 \leq N \leq 6$.

**Theorem 3.** *([18], [4], [8]) Let $3 \leq N \leq 6$. Let $u$ be a non-radial solution of (NLW), for $t \geq 0$. Assume that $\sup_{0<t<\infty} ||(u(t), \partial_t u(t))||_{\dot{H}^1\times L^2} < \infty$. Then, there exists $J \in \mathbb{N}\cup\{0\}$, a solution $v_L$ of the linear wave equation, a “well-chosen” sequence of times $t_n \to \infty$, and for each $j$, $1 \leq j \leq J$, a non-zero solution $Q^j$ of (7), a vector $\vec{l_j} \in \mathbb{R}^N, |\vec{l_j}| <$*

1, *scales* $\lambda_{j,n} > 0$, *positions* $x_{j,n} \in \mathbb{R}^N$*,such that*

$$u(t_n) = \sum_{j=1}^{J} (\lambda_{j,n})^{-\frac{N-2}{N}} Q^j_{\vec{l}_j}(x - x_{j,n}/\lambda_{j,n}, 0) + v_L(t_n) + o(1)$$

$$\partial_t u(t_n) = \sum_{j=1}^{J} (\lambda_{j,n})^{-\frac{N}{2}} \partial_t Q^j_{\vec{l}_j}(x - x_{j,n}/\lambda_{j,n}, 0) + \partial_t v_L(t_n) + o(1),$$

*with*

$$\frac{\lambda_{j,n}}{\lambda_{j',n}} + \frac{\lambda_{j',n}}{\lambda_{j,n}} + \frac{|x_{j,n} - x_{j',n}|}{\lambda_{j,n}} \underset{n\to 0}{\longrightarrow} 0,\ j \neq j',$$

*so that the traveling waves are "decoupled", and* $o(1)$ *is a term going to* $0$ *in* $\dot{H}^1$ *(respectively* $L^2$*), with* $n \to \infty$*.*

*Remark* 4. A method of proof relying on monotonicity after time average, as in Theorem 2 and Theorem 3, cannot give more than a decomposition for a "well-chosen" sequence of times. In fact, an example due to Topping [37] for the case of the geometric harmonic map heat flow shows this to be the case.

Moving forward, in trying to obtain versions of Theorem 2 and Theorem 3, which hold for all times, we can use the decompositions in those theorems as a starting point. We then need to see that the collision of two or more (well-separated) traveling waves or two (well-separated) static solutions, produce "radiation" in non-integrable situations, contrary to the Kruskal–Zabusky [40] numerics for the integrable cases. We have recently succeeded in proving this for the radial case of (NLW) for all odd dimensions $N$.

**Theorem 5.** *(*[9],[10],[11]*) The decomposition in Theorem 1 is valid for (NLW) in the radial case, in* $\mathbb{R}^N$*for all* $N$ *odd.*

## References


[1] T. Aubin, Equations diférentielles non lineaires et problème de Tamabe de Yamabe concernant la courbure scalaire, J. Math. Pures Appl. 55 (1976).

[2] M. Borghese, R. Jenkins and K.D.R. McLaughlin, Long time asymptotic behavior of the focusing nonlinear Schrödinger equation, Ann. Inst. H. Poincaré Anal. Non Linéaire 35 (2018), 887-920.

[3] J. D. Burchfield, Lord Kelvin and the age of the earth, Univ. of Chicago Press, 1990.

[4] T. Duyckaerts, H. Jia, C. Kenig and F. Merle, Soliton resolution along a sequence of times for the focusing energy critical wave equation, Geom. Funct. Anal. 27 (2017), 798-862.

[5] T. Duyckaerts, C. Kenig and F. Merle, Profiles of bounded radial solution of the focusing, energy-critical wave equation, Geom. Funct. Anal. 22 (2012), 639-698.

[6] T. Duyckaerts, C. Kenig and F. Merle, Classification of radial solutions of the focusing energy-critical wave equation, Cambridge J. of Math. 1 (2013), 75-144.

[7] T. Duyckaerts, C. Kenig and F. Merle, Profiles for bounded solutions of dispersive equations, with applications to energy-critical wave and Schrödinger equations, Comm. Pure Appl. Anal. 14 (2015), 1275-1326.

[8] T. Duyckaerts, C. Kenig and F. Merle, Scattering profile for global solutions of the energy-critical wave equation, J. Eur. Math. Soc. (JEMS) 21 (2019), 2117-2162.

[9] T. Duyckaerts, C. Kenig and F. Merle, Decay estimates for nonradiative solutions of the energy-critical focusing wave equation, arXiv:1912.07655, to appear, J. Geom. An.

[10] T. Duyckaerts, C. Kenig and F. Merle, Exterior energy bounds for the critical wave equation close to the ground state, arXiv: 1912.07658, to appear, Comm. Math. Phys.

[11] T. Duyckaerts, C. Kenig and F. Merle, Soliton resolution for the radial energy critical wave equation in all odd space dimensions, arXiv: 1912.07664.
[12] W. Eckhaus, The long time behavior for perturbed wave equations and related problems, in Trends in applications of pure mathematics to mechanics (Bad Honnef, 1985), vol 249 of Lecture Notes in Phys. Springer, Berlin, 1986, 168-194.
[13] W. Eckhaus and P. Schuur, The emergence of solitons of the Korteweg-de Vries equation from arbitrary initial conditions, Math. Methods Appl. Sci. 5 (1983), 97-116.
[14] E. Fermi, J. Pasta and S. Ulam, Los Alamos Report LA 1940, 1955.
[15] J. Fourier, Théorie analitique du chaleur, Firmin Didot, 1822.
[16] J. Fourier, The analytic theory of heat, Cambridge Univ. Press., 1878.
[17] B. Gidas, W. Ni, and L. Nirenberg, Symmetry of positive solutions of nonlinear elliptic equations in $\mathbb{R}^n$,in Mahtmeatical analysis and applications, Part A, vol. 7 of Adv. in Math. Suppl. Stud. Academic Press, New York, 1981, 369-402.
[18] H. Jia, Soliton resolution along a sequence of times with dispersive error for type II singular solutions to focusing energy critical wave equation, arXiv:1510.00075.
[19] H. Jia and C. Kenig, Asymptotic decomposition for semilinear wave and equivariant wave map equations, American Jour. of Math. 139 (2017), 1521-1603.
[20] C. Kenig and F. Merle, Global well-posedness, scattering and blow-up for the energy-critical, focusing, non-linear Schrödinger equation in the radial case, Invent. Math. 166 (2006), 645-675.
[21] C. Kenig and F. Merle, Global well-posedness, scattering and blow-up for the energy-critical, focusing, non-linear wave equation, Acta Math. 201 (2008), 147-212.
[22] C. Kenig and F. Merle, Scattering for $H^{1/2}$-bounded solutions to the cubic, defocusing nonlinear Schrödinger equation in 3 dimensions, Trans. Amer. Math. Soc. 362 (2010), 1937-1962.
[23] C. Kenig, G. Ponce and L. Vega, Well-posedness and scattering results for the generalized Korteweg-de Vries equation via the contraction principle, Comm. Pure Appl. Math. 46 (1993), 527-620.
[24] D.J. Korteweg and G. de Vries, On the change of form of long waves advancing in a rectangular canal, and on a type of long stationary waves, Philosophical Magazine 39 (1895), 422-443.
[25] M. Kruskal, The birth of the soliton, in nonlinear evolution equations solvable by the spectral transform, vol. 26 of Res. Notes in Math., Pitman, Boston Mass. 1978, 1-8.
[26] S. Pohozaev, On the eigenfunctions of the equation $\delta u + \lambda f(u) = 0$, Sov. Mat. Doklady 6 (1965), 1408-1411.
[27] C. Rodriguez, Profiles for the radial focusing energy-critical wave equation in odd dimensions, Adv. Differential Equations 21 (2016), 505-570.
[28] J. S. Russell, Report on waves, 1845, Report of the fourteenth meeting of the British Association fo the Advancement of Science, York, September 1844.
[29] P. Schuur, Asymptotic analysis of soliton problem, vol 1232 of Lecture Notes in Mathematics, Springer-Verlag, Berlin, 1986.
[30] H. Segur, Asymptotic solutions and conservation laws for the nonlinear Schrödinger equation II, J. Math. Phys. 17 (1976), 714-716.
[31] H. Segur and M. Ablowitz, Asymptotic solutions and conservation laws for the nonlinear Schrödinger equation I, J. Math. Phys. 17 (1976), 710-713.
[32] G. Talenti, Best constant in Sobolev inequality, Ann. Math. Pura Appl. 110 (1976), 353-372.
[33] T. Tao, A (concentration) compact attractor for high-dimensional non-linear Schödinger equations, Dyn. Part. Differ. Equ. 4 (2007), 1-53.
[34] T. Tao, A global compact attractor for high-dimensional defocusing non-linear Schrödinger equations with potential, Dyn. Part. Differ. Equ. 5 (2008), 101-116.
[35] T. Tao, Why are solitons stable? Bull. AMS 46 (2008), 1-33.
[36] W. Thompson, Letters on telegraph to America (1855), Math and Phys. papers vol 2, page 92.
[37] P. Topping, Rigidity in the harmonic map heat flow, J. Diff. Geom. 45 (1997), 593-610.
[38] S. Ulam, Adventures of a mathematician, University of California Press, Berkeley and Los Angeles, California, 1983.
[39] S. Wolfram, A new kind of science, Wolfram Media, 2002, page 407.

[40] N. Zabusky and M. Kruskal, Interactions of "solitons" in a collisionless plasma and the recurrence of initial states, Phys. Rev. Letters 15 (1965), 240-243.
[41] V. Zakharov, What is integrability? Springer Series in Nonlinear Dynamics, Springer-Verlag, Berlin, Heidelberg, 1991.
[42] V. Zakharov and A. Shabat, Exact theory of two-dimensional self-focusing and one-dimensional self-modulation of waves in nonlinear media. Z. Eksper. Teoret. Fiz. 61 (1971), 118-134.

UNIVERSITY OF CHICAGO
*Email address*: cek@math.uchicago.edu